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Holonomy in Quaternionic Quantum Mechanics

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The generalization of geometric phase for the quantum systems described by quaternionic quantum mechanics is given. The geometry of the quantum cyclic evolution is studied and the quaternionic Berry phase is shown to be given by the holonomy of the suitably defined fiber bundle.

KEY WORDS: Berry phase; holonomy; quaternionic quantum mechanics.

1. INTRODUCTION

The geometric ideas play an important role in physics. The quantum mechanical phenomena being geometric in its nature is the Berry phase (Anandan and Aharonov, 1990; Berry, 1984; Simon, 1983). After Berry's discovery the phase has been formalized in terms of the connection on the suitably defined fiber bundle as a holonomy. The topological nature of holonomy leads to the observation that the geometric phase is a global feature of the quantum evolution. In this paper the results are extended in the framework of the quaternionic quantum mechanics (Adler, 1995). The topological considerations allow to recognize the difference between results concerning triviality of the holonomy obtained in the standard complex and quaternionic quantum mechanics.

2. GEOMETRIC PHASE IN QUATERNIONIC QUANTUM MECHANICS

As a starting point, some concepts of quaternionic quantum mechanics will be introduced. Our approach is based on studies by Birkhoff and von Neumann, 1936; Finkelstein, Jauch, and Speiser, 1959; Adler, 1995. The space of states of the quantum system is a Hilbert space $\mathcal{H}(\mathbf{H})$ on the algebra of Hamilton's quaternions $\mathbf{H}$ with $\mathbf{H}$-valued scalar product $\langle \cdot \mid \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbf{H}$. The time evolution of the state is governed by a group of unitary operators in $\mathcal{H}$ generated by the antihermitian

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operator A satisfying the Schrödinger equation

$$\dot{\psi} = -A\psi. \tag{1}$$

Below we derive the formula for the geometric phase. The approach is a generalization of both Anandan's (Anandan, 1988) and Adler's approaches (Adler, 1995).

Let us consider the time evolution of the f-dimensional subspace of $\mathcal{H}$ with an orthonormal basis $\{\tilde{\psi}_a\}_{a=1}^{f}$. It evolves according to

$$\dot{\tilde{\psi}}_a = -A\tilde{\psi}_a, \quad a = 1 \ldots f. \tag{2}$$

Let us assume that there exists the set of f *cyclic states* $\{\psi_a\}_{a=1}^{f}$ such that $\psi_a(0) = \psi_a(T)$, $a = 1 \ldots f$ and for every $t \in [0, T]$ satisfying

$$\tilde{\psi}_a = \sum_{b=1}^{f} \psi_b U_{ba}(t) \tag{3}$$

with the unitary quaternionic matrix $U(t)$. Then the Schrödinger equation (1) implies

$$U_{ki}(T) = \exp\left[-\int_0^T \langle \psi_k \mid \dot{\psi}_i \rangle\, dt - \int_0^T \langle \psi_k \mid A\psi_i \rangle\, dt\right]. \tag{4}$$

The unitary transformation $U(T)$ is a matrix "phase factor," which is gained by the basis after a cyclic evolution. The term $g_{ki}(T) = \exp(-\int_0^T \langle \psi_k \mid \dot{\psi}_i \rangle\, dt)$ is a non-Abelian, nonadiabatic Berry phase and has, as will be shown, geometric origin.

3. GEOMETRY OF TIME EVOLUTION

Let E be a set of all f-dimensional subspaces of $\mathcal{H}$. We consider the fiber bundle $E = B \times \times G$ where the base space $B := E/G$ and $G = U(f)$ is a group of unitary quaternionic matrices. The bundle E is equivalent to the quaternion version of the bundle of the Stiefel manifold over the Grassman manifold with the fibre $U(f)$. For $f = 1$ it reduces to the Hopf bundle. Let $|\overleftarrow{n}\rangle \in B$ be represented by a diagonal matrix $|\overleftarrow{n}\rangle := diag(|n_1\rangle \cdots |n_f\rangle)$ with an orthonormal basis $\{|n_i\rangle\}_{i=1}^{f}$. Every $|\overrightarrow{n}\rangle \in E$, by the choice of a local section of the bundle E, can be written as $|\overrightarrow{n}\rangle = |\overleftarrow{n}\rangle_g$ with $g \in G$ and natural matrix multiplication. The right action of the group G on E is given by a natural matrix multiplication $R_g|\overrightarrow{n}\rangle := |\overrightarrow{n}\rangle g$. We define the dual $\langle\overleftarrow{n}|$ as a diagonal matrix formed by corresponding bra-vectors. The dual to $|\overrightarrow{n}\rangle \in E$ is a transposed matrix $\langle\overrightarrow{n}|$ with corresponding bra-vectors as elements. We define the multiplication of two elements of E (or B) to be a combination of the natural matrix multiplication and the scalar product in $\mathcal{H}$: $(\langle n_1 | \in_2\rangle)_{ij} := \sum_{k=1}^{f} \langle n_{1_{ij}} | n_{2_{kj}}\rangle$ or $(|n_1\rangle\langle n_2|)_{ij} := \sum_{k=1}^{f} |n_{1_{ik}}\rangle\langle n_{2_{kj}}|$. In the first case the result is a H-valued matrix while in the second the operator acting on E (or B).

Let $\mathcal{F}$ be the f-dimensional square matrix with every element $\mathcal{F}_{ij} = 1$. The one form $\omega(|\dot{\vec{n}}\rangle) := \langle\vec{n}|\mathcal{F}|d\vec{n}$ is a connection form on E.

The form ω is $\mathcal{G}$-valued where $\mathcal{G}$ is a Lie algebra of $G = U(f)$. Since $\langle\vec{n}|\mathcal{F}|\vec{n} = 1$ we obtain $(\omega(|\dot{\vec{n}}\rangle))^\dagger + \omega(|\dot{\vec{n}}) = 0$. The Lie algebra of G is known to satisfy (Nakahara, 1990) $\mathcal{G} = \{A \mid A^\dagger + A = 0\}$,which shows that ω is $\mathcal{G}$-valued. The one-form $\mathcal{G}$ satisfies $R_{g*}\omega(|\dot{\vec{n}}\rangle) = g^\dagger\omega(|\dot{\vec{n}})g$ and $\omega(A^\#) = \langle\vec{n}\,\exp tA|\mathcal{F}|\, d(\vec{n}\,\exp tA)\rangle$, where $A^\#|\dot{\vec{n}}\rangle = \frac{d}{dt}(|\vec{n})\,\exp tA)_{|t=0}$ is a fundamental field. It shows that ω is indeed a connection one-form (Nakahara, 1990).

Below we show that the geometric phase is a holonomy of a fiber bundle E. Let us consider *cyclic evolution* in B given by the loop $C : [0, T] \to B$ and $C(0) = C(T)$. There is a family of curves $\tilde{C} : [0, T] \to E$ such that $p\tilde{C}(t) = C(t)$ for every $t \in [0, T]$ where $p : E \to B$ is a bundle projection. If $\tilde{C}(0) = C(0)$ then $\tilde{C}(T) = \tilde{C}(0)_g(T)$ with a holonomy $g(T) \in G$ corresponding to the geometric phase. Let $\tilde{C}$ be the horizontal lift of C defined by $\omega(|\dot{\vec{n}}(t)\rangle) = 0$ for $\vec{n}(t) \in \tilde{C}$.

The equation for parallel transport defined by $\omega(|\vec{n}(t)\rangle) = 0$ or equivalently $\langle\vec{n}|\mathcal{F}|d\vec{n}\rangle = 0$ implies $0 = \langle\overleftarrow{n}(t)g(t)|\mathcal{F}|d(\overleftarrow{n}(t)g(t))\rangle$. Thus, the parallel transport along $\tilde{C}$ is described by the equation $\langle\overleftarrow{n}(t)|\mathcal{F}|d\overleftarrow{n}(t)\rangle + dg(t)g^\dagger = 0$. For $g(0) = 1 \in \mathcal{G}$ we obtain the holonomy $g(T) = \exp[-\int_0^T \langle\overleftarrow{n}(t)|\mathcal{F}|\frac{d}{dt}\overleftarrow{n}(t)\rangle\, dt]$ or, in an equivalent form

$$g(T) = \exp\left[-\oint_C \langle\overleftarrow{n}|\mathcal{F}|d\overleftarrow{n}\rangle\right]. \tag{5}$$

It is a formula describing the geometric (Berry) phase in (4). The expression $A(|\,\overleftarrow{n}\rangle) := \langle\overleftarrow{n}\,|\mathcal{F}|\,d\overleftarrow{n}\rangle$ is a local conection form on B.

The phase $g(T)$ does not depend on the time-independent gauge transformation, which is equivalent to the replacement of the section $|\,\overleftarrow{n}\rangle := |\,\overleftarrow{n}\rangle g(t)$ by $|\,\overleftarrow{n}\rangle := |\,\overleftarrow{n}\rangle gg(t)$ where g is t-independent. Writing $gg(t) =: \tilde{g}(t)$ we conclude that the holonomy (5) is preserved.

Further we present a simple application which follows from the existence of the connection one-form. We give a formula allowing to calculate a distance between two vectors in B in terms of vectors in E. Let us consider the identity (Anandan, 1990)

$$\frac{d}{dt}|\vec{n}\rangle = \frac{\delta}{dt}|\vec{n}\rangle + |\vec{n}\rangle\omega(|\dot{\vec{n}}\rangle)$$

where the first term is defined as a difference $\frac{d}{dt}|\vec{n}\rangle = \frac{\delta}{dt}|\vec{n}\rangle - |\vec{n}\rangle\omega(|\dot{\vec{n}}\rangle)$.

Definition of the connection form is equivalent to unique decomposition $T_qE = V_qE \oplus H_qE$ for every $q \in E$ of the tangent space T_qE into horizontal and vertical subspaces. For $q = |\vec{n}\rangle$ the vector $\frac{\delta}{dt}|\vec{n}\rangle \in H_qE$ since the horizontal subspace belongs to the kernel of ω. This vector is tangent to the curve $C : [0, T] \to B$ given by the projection $p|\vec{n}(t)\rangle = |\overleftarrow{n}(t)\rangle = C(t)$. It is implied by the fact that the vertical space belongs to the kernel of the tangent map induced by projection p. Thus $\frac{\delta}{dt}|\vec{n}\rangle = \frac{d}{dt}|\overleftarrow{n}\rangle$ for every curve $|\overleftarrow{n}(t)\rangle \subset B$.

Since $\frac{d}{dt}|\overleftarrow{n}\rangle = \frac{d}{dt}|\vec{n}\rangle - |\vec{n}\rangle\omega(|\dot{\vec{n}}\rangle) = \frac{d}{dt}|\vec{n}\rangle - |\vec{n}\rangle\langle\vec{n}|\mathcal{F}|\,\frac{d}{dt}|\vec{n}\rangle$ the integration yields $\int_{t_1}^{t_2} \frac{d}{dt}|\overleftarrow{n}\rangle\, dt = \int_{t_1}^{t_2}(\frac{d}{dt}|\vec{n}\rangle - |\vec{n}\rangle\langle\vec{n}|\mathcal{F}|\,\frac{d}{dt}\vec{n}\rangle)\, dt$. Thus

$$\Delta \equiv |\overleftarrow{n}(t_2)\rangle - |\overleftarrow{n}(t_1)\rangle = \int_{t_1}^{t_2} \left(\frac{d}{dt}|\vec{n}\rangle - |\vec{n}\rangle\langle\vec{n}|\mathcal{F}|\,\frac{d}{dt}\vec{n}\rangle\right) dt. \tag{6}$$

Taking an arbitrary norm of Δ gives a metric in B with a distance in base space B expressed in terms of vectors in E. The proposed metrics is purely geometric with no invocation of the specific evolution in opposition to the metrics proposed by Anandan, 1990.

4. TOPOLOGICAL ASPECTS OF QUANTUM HOLONOMY

In this section we analyze some topological properties of holonomy, which lead to nontrivial difference between quaternionic and complex Berry phase. We consider cyclic quantum evolution $f : M \to B$ where the parameter space $M \approx S^n$ is homeomorfic with n-dimensional sphere. It is well known that if $M = S^2$ the complex geometric phase is *nontrivial* in that sense that it cannot be removed by a globally defined gauge transformation (Szopa, 1992). It is caused by the fact that there is no globally defined coordinate system on the sphere and the local connection one-form is singular as in the case of $U(1)$ magnetic monopole. In the following we compare the triviality of the holonomy in quaternionic and complex quantum mechanics. We limit to the case when $f = 1$ and $E = B \times \times G$ is Hopf bundle: quaternionic (with a fiber topologically equivalent to S^3) or complex (with a fiber topologically equivalent to S^1). We call the geometric phase trivial if it is removable by a gauge transformation with no singularity, i.e. global. First we state without proof the theorem (Kiritsis, 1987):

Let $M \approx S^n$. For every loop $f : M \to B$ the geometric phase is trivial if and only if $\pi_n(E) = \pi_n(B)$, where $\pi_n(X)$ denotes the nth homotopy group of topological space X.

Let us consider the exact sequence of a bundle homotopy (Steenrod, 1951)

$$\pi_n(G) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(G) \tag{7}$$

In the quaternionic case for $f = 1$ the fiber $G \approx S^3$. Let us consider the case $n = 2$, i.e. the parameter space M homeomorfic with a two-dimensional sphere. Since both first and second homotopy group of S^3 is trivial, the sequence (7) yields

$$0 \to \pi_2(E) \to \pi_2(B) \to 0 \tag{8}$$

Since the sequence (8) is exact $\pi_2(E) = \pi_2(B)$, and according to the theorem stated above, the geometric phase is trivial contrary to the results obtained in complex quantum mechanics (Szopa, 1992). Therefore we showed that the global properties of quantum evolution can be different in the quantum mechanics over the algebra of quaternions and over the field of complex numbers.

5. CONCLUSIONS

Quantum mechanics with a Berry's discovery of the geometric phase has gained a new global aspect. This geometric and topological structure can also be considered in the framework of the quaternionc quantum mechanics. Geometric phase has been shown to be a holonomy of a suitably defined fiber bundle due to the existence of quaternionic analogy of the Anandan connection.

The topological analysis of the triviality of the geometric phase has shown the possible differences between results obtained in quaternionic and complex quantum mechanics. For the evolution generated by a parameter space topologically equivalent to the two-dimensional sphere the quaternionic geometric phase is trivial while the complex one not. The global properties of quaternionic quantum mechanics caused by the noncommutativity of Hamilton's quaternions can be different if compared with the complex case.

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